MATHEMATICS OF COMPUTATION
VOLUME 47, NUMBER 175
JULY 1986, PAGES 361-368

Computation of All the Amicable Pairs Below 10^{10}

By H. J. J. te Riele

Abstract. An efficient exhaustive numerical search method for amicable pairs is described. With the aid of this method all 1427 amicable pairs with smaller member below 10^{10} have been computed, more than 800 pairs being new. This extends previous exhaustive work below 10^8 by H. Cohen. In three appendices (contained in the supplements section of this issue), various statistics are given, including an ordered list of all the gcd's of the 1427 amicable pairs below 10^{10} (which may be useful in further amicable pair research). Suggested by the numerical results, a theorem of Borho and Hoffmann for constructing APs has been extended.

1. Introduction. Let $\sigma(m)$ denote the sum of all the divisors of m, including 1 and m. An *amicable* pair (AP) is a pair of positive integers (m, n), $m < n$, such that $\sigma(m) = \sigma(n) = m + n$. We note that m is *abundant* (since $\sigma(m) > 2m$) and that n is *deficient* (since $\sigma(n) < 2n$). The smallest AP is

$$(220, 284) = (2^2 5.11, 2^2 71).$$

In order to check whether or not a given positive integer m is the smaller member of an amicable pair, it seems necessary, at first sight, to compute $\sigma(m)$ and $n := \sigma(m) - m$, to check whether $n > m$ (i.e., whether m is abundant), and, if so, to compute $\sigma(n)$ and compare $\sigma(m)$ with $\sigma(n)$. This involves one or two complete factorizations, in case m is deficient or abundant, respectively. However, a closer look reveals that it is often possible to find out whether a given number m is deficient (hence cannot be the smaller member of an AP) without the need to factorize it completely. Moreover, once $\sigma(m)$ and n $(= \sigma(m) - m)$ have been computed, it is often possible to discover that $\sigma(n) \neq \sigma(m)$ without the need to factorize n completely.

These considerations have guided the design of an efficient exhaustive numerical AP search algorithm, the details of which are given in Section 2. With the aid of this algorithm we have extended Cohen's exhaustive list of all 236 APs with smaller member below 10^8 [4] to all 1427 APs with smaller member below 10^{10}. Of these, 601 have been published earlier [6], [7]. The other 826 seem to be new, and are published here for the first time (9 of them have been communicated to the author already in 1983 and 1984 by Woods (2), Borho (2) and Lee (5)). Section 3 presents details of the computations together with several tables collected from this search. Moreover, a result of Borho and Hoffmann for constructing APs is extended, as was suggested by the numerical tables.

Received March 27, 1985.
1980 *Mathematics Subject Classification*. Primary 10A40; Secondary 10-04.
Key words and phrases. Amicable pair.

Three appendices to this paper appear in the supplements section of this issue. These may also be obtained by writing to the author.

In Appendix I, we present the complete list of all 1427 APs with smaller member below 10^{10} ordered according to the size of the smaller members of the pairs. Appendix II displays the same list with a different ordering, viz., according to the various occurring types (defined in Section 3). Finally, Appendix III tabulates all the greatest common divisors of the 1427 APs, in increasing order, together with their frequencies of occurrence, and, for each gcd g, the rank numbers of all the APs (m, n) for which $\gcd(m, n) = g$.

2. Check Whether a Given m is the Smaller Member of an AP. Let p_i be the ith prime, $P_{ij} := \prod_{k=i}^{i+j-1} p_k$, $Q_{ij} := \prod_{k=i}^{i+j-1} p_k/(p_k - 1)$. We start with the following lemma which gives an upper bound for $\sigma(m)/m$.

LEMMA 2.1. *If m only has prime divisors $\geqslant p_i$ $(i \geqslant 1)$ and if $m < P_{i,j+1}$ $(j \geqslant 1)$ then $\sigma(m)/m < Q_{ij}$.*

Proof. Since $m < P_{i,j+1} = p_i p_{i+1} \cdots p_{i+j}$, and since any prime divisor of m is $\geqslant p_i$, it follows that m has at most j different prime divisors $\geqslant p_i$ (otherwise we would have $m \geqslant p_i p_{i+1} \cdots p_{i+j} = P_{i,j+1}$). This implies that

$$\frac{\sigma(m)}{m} = \prod_{p^e \| m} \frac{p^{e+1} - 1}{p^e(p-1)} = \prod_{p^e \| m} \frac{p - p^{-e}}{p - 1} < \prod_{p | m} \frac{p}{p-1} \leqslant \prod_{k=i}^{i+j-1} \frac{p_k}{p_k - 1} = Q_{ij}. \quad \square$$

In the algorithm below, this lemma is invoked very frequently. Therefore, we require a precomputed table of P- and Q-values, large enough so that the values needed can be found quickly by simple table look-ups.

Now we describe an efficient algorithm to check whether a given positive integer m belongs to an AP (m, n) with $m < n$. This algorithm is based on the observation that when, for given γ and N, we want to verify one of the relations $\sigma(N)/N > \gamma$, $= \gamma$, $< \gamma$, and when the primes $2, 3, \ldots, p$ have been tried as divisors of N, it may be possible

(i) to detect, with Lemma 2.1, whether $\sigma(N)/N < \gamma$ by using the information that the *unfactored* portion of N only has prime divisors $> p$, and

(ii) to detect whether $\sigma(N)/N > \gamma$ by using the *factored* portion of N.

In this way, much unnecessary factorization time may be avoided. The price to pay for this gain lies in the time needed to consult the P- and Q-tables used in Lemma 2.1. In the algorithm, the index $i_{\max}$ is the maximum value of i for which Lemma 2.1 is invoked. In order to restrict this table look-up time, $i_{\max}$ should not be chosen too large. The optimal value of $i_{\max}$ also depends on the actual implementation of the algorithm (cf. Section 3).

Algorithm to Check Whether m is the Smaller Member of an AP.

Step 1. (Find out whether m is abundant; in this step, keep $m = m_1 m_2$ where $\gcd(m_1, m_2) = 1$, m_1 is the factored and m_2 is the unfactored portion of m, $\alpha := \sigma(m_1)/m_1$; start with $m_1 := 1$, $m_2 := m$, $\alpha := 1$.)

Start factoring m by trial dividing m_2 by the primes $p_1, p_2, \ldots \leqslant m_2^{1/2}$. In case a prime power divisor p_{i-1}^e $(e \geqslant 1)$ of m_2 has been found, update m_1, m_2 and α $(m_1 := m_1 p_{i-1}^e, m_2 := m/m_1, \alpha := \alpha \cdot \sigma(p_{i-1}^e)/p_{i-1}^e)$. After the trial division with p_{i-1} (whether or not p_{i-1} divides m_2): if $\alpha < 2$ and $4 \leqslant i \leqslant i_{\max}$, check whether m

is possibly deficient as follows: by inspecting the P-table find the smallest value of j $(=:j^*)$ such that $m_2 < P_{i,j+1}$; if $\alpha Q_{i,j^*} < 2$, then STOP (because, in that case, m is deficient: by Lemma 2.1 we have $\sigma(m_2)/m_2 < Q_{i,j^*}$ so that

$$\frac{\sigma(m)}{m} = \frac{\sigma(m_1)}{m_1} \cdot \frac{\sigma(m_2)}{m_2} = \alpha \frac{\sigma(m_2)}{m_2} < \alpha Q_{ij^*} < 2).$$

If $\alpha \geqslant 2$, or $i < 4$ or $i > i_{\max}$, the deficiency check on m is left out. After the complete factorization of m (and simultaneous computation of $\sigma(m)$): if $m < \sigma(m) - m =: n$ (i.e., m is abundant), go to Step 2, otherwise STOP.

End of Step 1

Step 2. (Given m, $\sigma(m)$ and $n = \sigma(m) - m$, check whether $\sigma(n) = \sigma(m)$; during the factorization of n try to exclude those m for which $\sigma(n) \neq \sigma(m)$ as early as possible by testing whether $\sigma(n)/n \neq \beta$ where $\beta = \sigma(m)/n$; in this step, keep $n = n_1 n_2$, where $\gcd(n_1, n_2) = 1$, n_1 is the factored and n_2 the unfactored portion of n, $\alpha := \sigma(n_1)/n_1$; start with $n_1 := 1$, $n_2 := n$, $\alpha := 1$.)

Start factoring n by trial dividing n_2 by the primes $p_1, p_2, \ldots \leqslant n_2^{1/2}$. In case a prime power divisor p_{i-1}^e $(e \geqslant 1)$ of n_2 has been found, update n_1, n_2 and α: if the updated α satisfies $\alpha > \beta$, then STOP (because, in that case, we have

$$\frac{\sigma(n)}{n} = \frac{\sigma(n_1)}{n_1} \frac{\sigma(n_2)}{n_2} \geqslant \frac{\sigma(n_1)}{n_1} = \alpha > \beta = \frac{\sigma(m)}{n},$$

so that $\sigma(n) \neq \sigma(m)$). After the trial division with p_{i-1} (whether or not p_{i-1} divides n_2): if $4 \leqslant i \leqslant i_{\max}$ check whether $\sigma(n)/n < \beta$ as follows: by inspecting the P-table find the smallest value of j $(=:j^*)$ such that $n_2 < P_{i,j+1}$. If $\alpha Q_{i,j^*} < \beta$, then STOP (because, in that case, $\sigma(n)/n < \beta$: by Lemma 2.1 we have $\sigma(n_2)/n_2 < Q_{i,j^*}$ so that

$$\frac{\sigma(n)}{n} = \frac{\sigma(n_1)}{n_1} \cdot \frac{\sigma(n_2)}{n_2} = \alpha \frac{\sigma(n_2)}{n_2} < \alpha Q_{i,j^*} < \beta).$$

If $i < 4$ or $i > i_{\max}$, the check on $\sigma(n)/n < \beta$ is omitted. After the complete factorization of n (and simultaneous computation of $\sigma(n)$): check whether $\sigma(n) = \sigma(m)$. If so, (m, n) is an AP.

End of Step 2

3. Computing All the APs Below 10^{10}. In order to compute all the APs (m, n) with $m < n$ and $10^8 < m \leqslant 10^{10}$ (thus extending H. Cohen's computations reported in [4]), we distinguish between $m \equiv 0 \pmod 6$ (the easy case), and $m \not\equiv 0 \pmod 6$ (the hard case).

If $m \equiv 0 \pmod 6$ and $n = \sigma(m) - m$ is *even*, then (m, n) cannot be an AP [5]. Therefore, n should be *odd*. In that case, we have [6] $m = 2^\mu M^2$, $n = N^2$, with $\mu \in \mathbb{N}$, M and N being odd. For all the numbers $m = 2^\mu M^2$ with $3 \mid M$ and $10^8 < m \leqslant 10^{10}$, we computed $n := \sigma(m) - m$ and checked whether n was a perfect square. Not a single such case was found. Computer time was about 6 CPU seconds.

For all $m \not\equiv 0 \pmod 6$ with $10^8 < m \leqslant 10^{10}$ we used the algorithm of Section 2 to find all APs in this range. The optimal choice of $i_{\max}$ for our FORTRAN-implementation on a CYBER 750 was about 75. This value was chosen to be fixed for the whole range. The speed-up factor of our program was about 15, compared with a straightforward program which, given m, computes $\sigma(m)$ and, if $n := \sigma(m) - m > m$, computes $\sigma(n)$. A slight increase of the speed was obtained as follows. In Step 1, in case a prime (power) factor of m_2 was found and m_1 and $\sigma(m_1)$ (among others) were updated, it was checked whether *both* m_1 *and* $\sigma(m_1)$ were divisible by one of the primitive abundant numbers $20 = 2^2 5$, $28 = 2^2 7$, $70 = 2.5.7$ and $88 = 2^3 11$. If so, the algorithm was stopped since this implied that also m and $\sigma(m)$, hence also $n = \sigma(m) - m$ were divisible by this abundant number, so that both m and n were abundant. This is impossible for an AP (m, n).

The total time to cover the range $10^8 < m \leqslant 10^{10}$ was about 1000 (low priority) CPU hours, spent in the last seven months of 1984.

The total number of APs (m, n) found with $m < n$ and $10^8 < m \leqslant 10^{10}$ was 1191. In Appendix I (of the supplements section) all the APs with smaller member $\leqslant 10^{10}$ are given (including the 236 APs with smaller member $\leqslant 10^8$). For each pair we list the decimal representation and the prime factorization of the members, a rank number, a code (letter plus digit) referring to the discoverer, and the type of the pair (defined below). For example, pair #1427 reads as follows:

1427	9967523980	2E2.257.5.17.37.3083
R9 42	12890541236	2E2.257.107.117191.

Table 1 gives the meaning of the codes, and their frequencies of occurrence. Extensive information about the sources of the pairs with code L1 is given in the survey paper [6].

There are 1015 pairs with even members and 412 with odd members. The minimal and maximal values of m/n are 0.6979 and 0.999858 for the APs #567 and #1010, respectively.

Let $A(x)$ be the number of APs (m, n) with $m < n$ and $m \leqslant x$. From the list of APs with $m \leqslant 10^8$, Bratley et al. [3] concluded that for $x \leqslant 10^8$, $A(x)$ is approximately proportional to $x^{1/2}/\ln(x)$. In Table 2 we give, for $x = k.\,10^9$ $(1 \leqslant k \leqslant 10)$: $A(x)$, $A(x)\ln(x)/x^{1/2}$, $A(x)(\ln(x))^2/x^{1/2}$ and $A(x)(\ln(x))^3/x^{1/2}$. From these figures we may draw the conclusion that for $x \leqslant 10^{10}$, $A(x)$ is approximately proportional to $x^{1/2}/(\ln(x))^3$.

TABLE 1

Status list of the first 1427 *APs* (m, n), $m < n$, *with* $m \leqslant 10^{10}$

code	#APs	references and remarks
L1	508	[6]
R2	1	[9] (#1056)
W1	73	sent to the author by D. Woods on June 29, 1982 and published in [7]
R3	19	found by the author with the methods described in [8], and published in [7]
W2	1	sent in by D. Woods on Feb. 16, 1983 (#330)
R6	1	found by the author in May, 1983 (#1375)
W3	1	sent in by D. Woods on July 11, 1983 (#1050)
L2	5	sent in by E. J. Lee in July, 1984 (##778, 860, 894, 1241, 1261)
B4	2	sent in by W. Borho on Nov. 2, 1984 (##809, 1393)
R9	816	found by the author during the systematic search described in this paper

TABLE 2

Comparison of $A(x)$ with $x^{1/2}/(\ln(x))^i$, $i = 1,2,3$

$x/10^9$	$A(x)$	$A(x)\ln(x)/x^{1/2}$	$A(x)(\ln(x))^2/x^{1/2}$	$A(x)(\ln(x))^3/x^{1/2}$
1	586	0.3840	7.958	164.9
2	762	0.3649	7.815	167.4
3	898	0.3578	7.807	170.4
4	1009	0.3527	7.799	172.4
5	1100	0.3474	7.759	173.3
6	1185	0.3444	7.755	174.6
7	1256	0.3403	7.715	174.9
8	1317	0.3358	7.656	174.6
9	1377	0.3327	7.625	174.8
10	1427	0.3286	7.566	174.2

We define an AP (m, n), $m < n$, to be a *regular amicable pair of type* (i, j), if $(m, n) = (gM, gN)$, where $g = \gcd(m, n)$, $\gcd(g, M) = \gcd(g, N) = 1$, M and N are squarefree, and the numbers of prime factors of M and N are i and j, respectively. Other pairs are called *irregular* or *exotic*. There are 1082 regular and 345 irregular APs with smaller member $\leqslant 10^{10}$. It is easy to see that there are no regular pairs of type $(1, j)$, $j \geqslant 1$: let g be the gcd of such an AP, so that $(m, n) = (gp, gN)$ where p is a prime and $\gcd(g, p) = \gcd(g, N) = 1$. We have $m < n$, hence $p < N$. By definition, $\sigma(gp) = \sigma(gN)$, implying that $p + 1 = \sigma(N)$. Since, for any $N \in \mathbb{N}$, $\sigma(N) > N$, this implies that $p + 1 > N$, a contradiction. We note that in this argument N need not be squarefree.

In Table 3 we give the frequency distribution of the various types among the first 1082 regular APs. We note that there are relatively few regular APs of type $(i, 1)$, $i \geqslant 2$, and of type (i, j) with $i < j$.

In [7] the total number of known APs with smaller member $\leqslant 10^{10}$ was 601 (these are the APs belonging to the first four codes in Table 1). Among them were 104 irregular APs, i.e., 17.3%. Comparing this figure with the 345 irregular APs in our *complete* list of APs with smaller member $\leqslant 10^{10}$, i.e., 24.2%, we see that relatively many irregular APs were found in our systematic search.

In Appendix II (of the supplements section) we present lists of all the 1082 regular APs arranged according to their types, together with a list of the 345 exotic APs. This appendix may be useful for searches of APs of a special type.

The regular pairs of type $(i, 1)$, $i \geqslant 2$, play an important role as "mother" pairs in methods to generate new APs from given pairs. In [8] a substantial part of the new APs found there was constructed from such mother pairs. In [1], Borho and Hoffmann have partially generalized the methods from [8] by introducing the concept of a *breeder*: a breeder is a pair of positive integers (a_1, a_2) such that the equations

$$a_1 + a_2 x = \sigma(a_1) = \sigma(a_2)(x + 1)$$

TABLE 3

Frequency distribution of the first 1082 *regular APs of type* (i, j), $i \geqslant 2$, $j \geqslant 1$

$i =$	$j =$ 1	2	3	4	5	row totals
2	20	67	21	4	0	112
3	16	271	280	24	0	591
4	1	78	201	63	2	345
5	0	6	18	7	3	34
column totals	37	422	520	98	5	1082

have a positive integer solution x. If x is a prime, then (a_1, a_2x) is an amicable pair. For certain breeders, called "special" breeders, Borho and Hoffmann formulate the following

THEOREM 1 [1]. *Let* (a_1, a_2) *be a special breeder, i.e.,* $a_1 = au$, $a_2 = a$, *with* $\gcd(a, u) = 1$. *Take any factorization of* $C := \sigma(u)(u + \sigma(u) - 1)$ *into two different factors* D_1, D_2 $(C = D_1D_2)$. *Then, if the numbers* $s_i = D_i + \sigma(u) - 1$, *for* $i = 1, 2$, *and also* $q = u + s_1 + s_2$ *are primes not dividing* a, *then* (auq, as_1s_2) *is an amicable pair.* □

Regular APs of type $(i, 1)$, $i \geqslant 2$, are of the form (au, ap), p prime, and the numbers (au, a) are special breeders which generally produce many APs with the above theorem.

In our list of 1427 APs we found a few APs, e.g., #647 and #955, which suggested that the condition $\gcd(a, u) = 1$ in Theorem 1 may be dropped. In fact, we have

THEOREM 2. *Let* (au, a) *be a breeder, i.e., there exists a positive integer* x *such that* $au + ax = \sigma(au) = \sigma(a)(x + 1)$. *Take any factorization of* $C := (x + 1)(x + u)$ *into two different factors* D_1, D_2 $(C = D_1D_2)$. *Then, if the numbers* $s_i = D_i + x$, *for* $i = 1$, 2, *and also* $q = u + s_1 + s_2$ *are primes not dividing* a, *then* (auq, as_1s_2) *is an amicable pair.* □

The proof of this theorem is left to the reader.

If $\gcd(a, u) = 1$, then $\sigma(au) = \sigma(a)\sigma(u)$, so that $x = \sigma(u) - 1$ and Theorem 2 reduces to Theorem 1. As an example, AP #955 gives the breeder (au, a) with $a = 3.5.7.19$ and $u = 7.29.47.181$. Theorem 2 yields 16 new APs with this breeder as input.

It is known [5] that most even APs have a pair sum which is $\equiv 0 \pmod 9$. Our search proves that indeed Poulet's pair #503: $(2^4 331.19.6619, 2^4 331.199.661)$ is the smallest exceptional pair. All known exceptional pairs had members $\equiv 7 \pmod 9$ and a pair sum $\equiv 5 \pmod 9$. In our search, we found two even APs with pair sum $\equiv 3 \pmod 9$, viz., the (irregular) pairs:

$$\#577\colon 2^4\begin{cases} 19^2 103.1627 \\ 3847.16763 \end{cases} \quad \text{and} \quad \#874\colon 2^2 19\begin{cases} 13^2 37.43.139 \\ 41.151.6709. \end{cases}$$

TABLE 4

The 17 *APs among the first* 1427, *whose pair sum is* $\not\equiv 0 \pmod 9$

	even members	odd members
regular	#503, type (2,2) #1031, type (2,2) #1081, type (2,2)	#899, type (3,2) #1057, type (2,2) #1158, type (3,2)
irregular	# #577, 874	# #7, 38, 78, 113, 256, 440, 1083, 1175, 1380

TABLE 5

All (37) *pairs from the first* 1427 *APs having the same pair sum*

rank numbers		pair sum	prime decomposition of the pair sum, i.e., exponents belonging to the primes											
			2	3	5	7	11	13	17	19	23	29	31	37
32	35	1296000	7	4	3									
105	109	20528640	9	6	1		1							
137	138	37739520	10	4	1	1		1						
172	173	75479040	11	4	1	1		1						
272	276	321408000	10	4	3								1	
282	286	348364800	13	5	2	1								
350	351	556839360	6	6	1	1	1						1	
347	355	579156480	9	5	1	2				1				
373	375	638668800	12	4	2	1	1							
368	377	661893120	12	5	1	1				1				
395	399	761177088	10	5		1				1	1			
411	415	796340160	6	5	1	2	1			1				
427	433	883872000	8	4	3		1						1	
462	476	1181174400	7	5	2	2							1	
486	491	1282417920	8	5	1	1				1			1	
574	582	2068416000	9	5	3	1				1				
626	630	2395008000	10	5	3	1	1							
653	665	2682408960	12	5	1	2	1							
695	697	3155023872	11	4		1	1	1		1				
717	730	3599769600	13	4	2	1							1	
751	753	4049740800	10	6	2	1							1	
798	807	4606156800	13	3	2	2			1					
786	787	4716601344	13	2		1		1		1				1
824	840	5094835200	10	7	2	1		1						
940	941	6824563200	9	3	2	2		1					1	
926	952	6897623040	13	7	1	1	1							
997	998	7925299200	11	5	2	2		1						
1012	1019	8273664000	11	5	3	1				1				
1069	1097	10027929600	12	5	2			1					1	
1124	1142	11195712000	9	3	3		1			1			1	
1147	1150	11416204800	9	4	2	1	2	1						
1143	1181	12098211840	12	5	1		1	1	1					
1232	1233	13473008640	10	5	1	2		1	1					
1254	1265	14341017600	12	4	2	1		1		1				
1249	1255	14478912000	9	5	3	2				1				
1272	1278	15058068480	10	5	1	2		1		1				
1410	1425	19926466560	14	5	1	1	1	1						

These are the first two examples of APs of the form described in [5, Theorem I, case (b)] (also cf. the remarks immediately following Table I in [5]). Table 4 gives the rank numbers of the 17 APs with smaller member $\leqslant 10^{10}$ whose pair sum is $\not\equiv 0 \pmod 9$, divided into even and odd pairs, and regular and irregular pairs.

Another question, suggested by Professor C. Pomerance, is whether pairs, triples, quadruples, etc. of APs exist having the *same pair sum*. Among the first 1427 APs, we found 37 such pairs of APs, but no such triples, quadruples, etc. Table 5 gives the rank numbers of these pairs of APs, and the prime factorization of their pair sums. The pair sums only have prime divisors $\leqslant 37$. In 30 of the 37 cases at least one member of the pair was found during the exhaustive search described in the present paper.

In Appendix III (of the supplements section) we tabulate all the greatest common divisors of the first 1427 APs, ordered according to their size, with frequencies, and with the rank numbers of all the APs corresponding to a given gcd. This might be useful in further searches for special APs, and in searches for so-called *isotopic* APs (cf., [6, p. 83]). For example, new APs, isotopic with APs from the list of 1427 APs, are obtained by replacing the common factor $3^3 5$ in ##882 and 1087 by $3^2 7.13$, by replacing the common factor $3^3 5^3$ in #1205 by $3^2 5^2 31$, and by replacing the common factor $3^3 5^2 31$ in ##717 and 1228 by $3^6 5.23.137.547.1093$, and by $3^{10} 5.23.107.3851$.

In [8], we have presented methods to find new APs from known APs. By applying these methods to the new APs among the first 1427 APs, we have found 117 new APs (with smaller member $> 10^{10}$). The new APs were found mainly from mother pairs having a relatively simple structure, like those of type $(i, 1)$, $i > 1$. They will be published in a forthcoming report [2], together with many other new amicable pairs.

Centre for Mathematics and Computer Science
Kruislaan 413
1098 SJ Amsterdam
The Netherlands

1. W. BORHO & H. HOFFMANN, "Breeding amicable numbers in abundance," *Math. Comp.*, v. 46, 1986, pp. 281–293.

2. W. BORHO, H. HOFFMANN & H. J. J. TE RIELE, *Table of Amicable Pairs Between* 10^{10} *and* 10^{52}, CWI-report. (In preparation.)

3. P. BRATLEY, F. LUNNON & JOHN MCKAY, "Amicable numbers and their distribution," *Math. Comp.*, v. 24, 1970, pp. 431–432.

4. H. COHEN, "On amicable and sociable numbers," *Math. Comp.*, v. 24, 1970, pp. 423–429.

5. E. J. LEE, "On divisibility by nine of the sums of even amicable pairs," *Math. Comp.*, v. 23, 1969, pp. 545–548.

6. E. J. LEE & J. S. MADACHY, "The history and discovery of amicable numbers," *J. Recreational Math.*, v. 5, 1972; Part I: pp. 77–93, Part II: pp. 153–173, Part III: pp. 231–249.

7. H. J. J. TE RIELE, *Table of* 1869 *New Amicable Pairs Generated from* 1575 *Mother Pairs*, Report NN 27/82, Math. Centre, Amsterdam, Oct. 1982.

8. H. J. J. TE RIELE, "On generating new amicable pairs from given amicable pairs," *Math. Comp.*, v. 42, 1984, pp. 219–223.

9. H. J. J. TE RIELE, *Further Results on Unitary Aliquot Sequences*, Report NW 2/78, Math. Centre, Amsterdam, 2nd ed., Jan. 1978.